数理解析研究所講究録
619巻 1987年 80-96

On the classification of Legendre immersions *)

Masahisa ADACHI and Kenji YAMATO
(足立正久 京大理) (大和健二 阪大教養)

0. Introduction

In this paper we shall give a homotopy-theoretic classification of the Legendre immersions $\Lambda \to M$ of a smooth n-manifold Λ into a regular contact manifold M of dimension $2n + 1$. Recall that a contact structure σ on M is a differential 1-form on M with $\sigma \wedge \underbrace{d\sigma \wedge \ldots \wedge d\sigma}_{n} \neq 0$ at each point $x \in M$. A smooth immersion $\lambda : \Lambda \to M$ is called a Legendre immersion if σ vanishes on each vectors in $T(M)$ tangent to $\lambda(\Lambda)$, that is, if $\lambda^* \sigma = 0$. To each Legendre immersion $\lambda : \Lambda \to M$ we can associate its differential $d\sigma : T(\Lambda) \to T(M)$. By definition $d\sigma$ is a monomorphism which takes each fibre $T_p(\Lambda)$ to a Legendre plane in $T_{\lambda(p)}(M)$, that is, an n-plane on which σ vanishes. We call such monomorphism L-monomorphisms, so that d sends each Legendre immersion to an L-monomorphism.

We will say that two Legendre immersions λ_0 and λ_1 are L-regularly homotopic, if there is a smooth regular homotopy λ_t between λ_0 and λ_1, such that λ_t is a Legendre immersion

*) Dedicated to Professor Nobuo Shimada on his 60th birthday

for each t. Similarly, we can speak of a homotopy through L-monomorphisms, of $T(\Lambda)$ into $T(M)$.

The following is the main theorem of this paper.

Theorem 1. Let Λ be a simply connected smooth n-manifold and M be a compact regular contact $(2n + 1)$-manifold. Then d induces a 1-1 correspondence between L-regular homotopy classes of L-immersions $\Lambda \longrightarrow M$ and L-homotopy classes of L-monomorphisms $T(\Lambda) \longrightarrow T(M)$.

The concept of regular contact manifolds was introduced by Boothby and Wang [4]. We recall this in section 1. Whether for an arbitrary smooth n-manifold Λ and an arbitrary contact $(2n + 1)$-manifold M the theorem above still holds or not, is open.

This paper is motivated by Bennequin [3], Douady [5].

Our approach is inspired by Gromov [6] and Lees [11].

1. Regular contact manifolds

We recall here regular contact manifolds.

Let $M = (M, \sigma)$ be a contact manifold of dimension $2n + 1$. Namely, M is a C^{∞}-manifold of dimension $2n + 1$ and σ be a differential 1-form on M with $\sigma \wedge (d\sigma)^n \neq 0$ at each point $x \in M$, where $(d\sigma)^n = d\sigma \wedge \ldots \wedge d\sigma$.

A quadratic form θ of the Grassman algebra $\wedge V^*$, where

V^* is the dual to a vector space V, is said to have <u>rank</u> $2r$, if the exterior product $(\theta)^r \neq 0$ but $(\theta)^{r+1} = 0$. Equivalently rank θ = dim V - dim V_0, where $V_0 = \left\{ X \in V \,\middle|\, \theta(X, V) = 0 \right\}$. It follows that on a contact manifold M the condition $\sigma \wedge (d\sigma)^n \neq 0$ implies that at each point $x \in M$ the quadratic form $d\sigma$ in the Grassman algebra $\wedge T_x^*(M)$ has rank $2n$. We then have

$$V_0 = \left\{ X \in T_x(M) \,\middle|\, d\sigma(X, T_x(M)) = 0 \right\}$$

is a subspace of dimension one on which $\sigma \neq 0$, and which is thus complementary to the $2n$-dimensional subspace on which $\sigma = 0$.

Let Z_x be the element of V_0 on which σ has the value 1. Then Z is a vector field, which we call <u>associated</u> to σ, defined on all of M by σ, and which is never zero since $\sigma(Z) = 1$. This vector field defines an involutive differential system on M and we shall call the contact structure σ <u>regular</u> if each point has a <u>regular neighborhood</u>, i.e. a cubical coordinate neighborhood $(x_1, \dots, x_{2n+1})$ where intersection with any given integral curve corresponds to a single segment

$$x_2 = c_2, \dots, x_{2n+1} = c_{2n+1}, \quad c_i = \text{constant}, \; i = 2, \dots, 2n+1,$$

i.e., which is thus pierced at most once by any given integral curve. this implies in particular that each integral curve is a closed point set.

Hereafter, we will assume the manifold M to be compact. If σ is a regular contact form on M, then, since Z is never zero and since the integral curves are closed and thus compact set, we see that they must be homeomorphic to the circle S^1. Moreover, the vector field Z generates a global action of the additive group of real numbers R on M. It is clear from the above that we may suppose that the associated vector field Z generates an action of the circle group S^1 on M. If B denotes the set of orbits, it follows that B is a C^∞-manifold, and that if $(u_1, \ldots, u_{2n+1})$ is a regular coordinate neighborhood in M, the orbit corresponding to u_2 = constant, ... , u_{2n+1} = constant, then $U' = p(U)$ with coordinates $u_2, \ldots, u_{2n+1}$ is a coordinate neighborhood on B, where $p : M \longrightarrow B$ is the natural projection.

Boothby and Wang [4] proved the following theorem.

Theorem 1.1. If σ is a regular contact form on a compact manifold M, then

(i) M is a principal bundle over B with group and fibre S^1,

(ii) σ defines a connection in this bundle,

(iii) the base space B is a symplectic manifold whose symplectic structure ω determines an integral cocycle on B and is the curvature form of σ, i.e. $d\sigma = p^*\omega$ is the equation of structure of the connection.

Actually ω is the characteristic class (with real coefficients) of the circle bundle M.

2. Spaces of Legendre immersions

Let Λ be a smooth n-manifold and M be a contact $(2n + 1)$-manifold with contact structure σ.

In order to prove Theorem 1, we consider the space $L\text{-Imm}(\Lambda, M)$ of all Legendre immersions of Λ in M with C^∞-topology.

Let $L\text{-Mon}(T(\Lambda), T(M))$ denote the space of all L-monomorphisms of $T(\Lambda)$ into $T(M)$ with compact-open topology.

Observe that the differential d defines a map

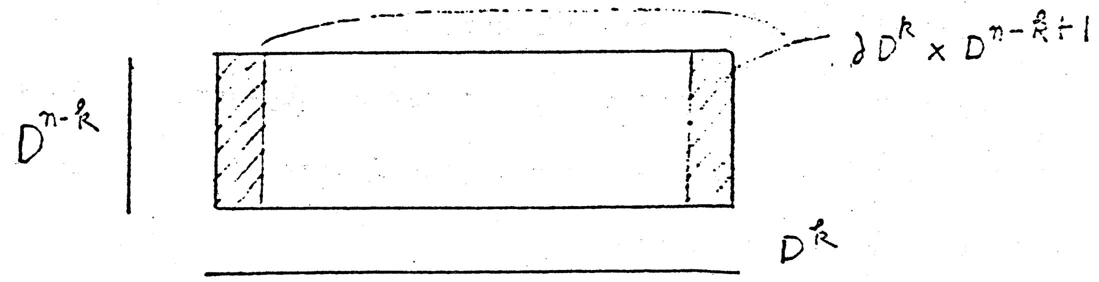
$$d : L\text{-Imm}(\Lambda, M) \longrightarrow L\text{-Mon}(T(\Lambda), T(M)).$$

We shall prove the following theorem.

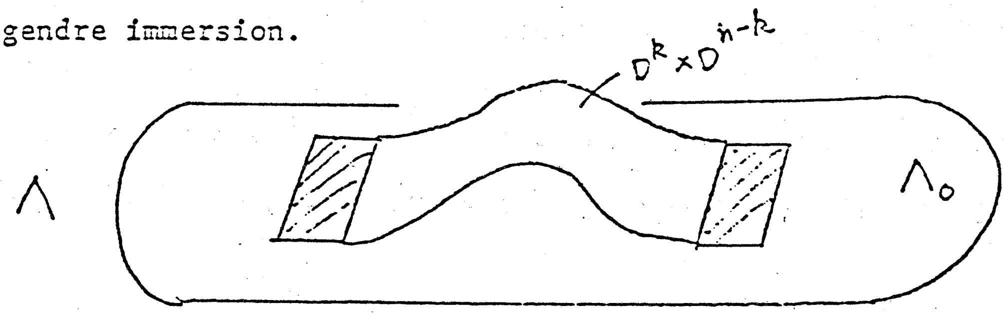
Theorem 2.1. Let Λ be a simply connected smooth n-manifold and M be a compact regular contact $(2n + 1)$-manifold. Then the map $d : L\text{-Imm}(\Lambda, M) \longrightarrow L\text{-Mon}(T(\Lambda), T(M))$ is a weak homotopy equivalence.

Theorem 1 follows directly from Theorem 2.1 : since d is a weak homotopy equivalence, it induces a 1-1 correspondence $d_* :$ $\pi_0(L\text{-Imm}(\Lambda, M)) \longrightarrow \pi_0(L\text{-Mon}(T(\Lambda), T(M)0$ of path components, that is, of L-regular homotopy classes of Legendre immersions and homotopy classes of L-monomorphisms.

The first step in the proof of Theorem 2.1 is to establish a covering homotopy for spaces of Legendre immersions. For convenience of notations, we will denote the p-cube by I^p. Write D^k for standard k-disk in R^k, and $D^k \times D^{n-k+1}$ for a neighborhood of $\partial D^k \times D^{n-k}$ in $D^k \times D^{n-k}$.



Let $\Lambda = \Lambda_0 \cup (D^k \times D^{n-k})$, where $\Lambda_0 \cap (D^k \times D^{n-k}) = \partial D^k \times D^{n-k+1}$. Let $f : \Lambda \longrightarrow M$, and suppose that $f|\Lambda_0$ is a Legendre immersion.



Theorem 2.2. Let $\pi : \text{L-Imm}(D^k \times D^{n-k}, M) \longrightarrow \text{L-Imm}(\partial D^k \times D^{n-k+1}, M)$ be the map which maps f to $f|\partial D^k \times D^{n-k+1}$. Let $F_0 : I^p \longrightarrow \text{L-Imm}(D^k \times D^{n-k}, M)$, $F : I^p \times I \longrightarrow \text{L-Imm}(\partial D^k \times D^{n-k+1}, M)$ be continuous maps such that $\pi \circ F_0(x) = F(x, 0)$. Then there exists a continuous map $\tilde{F} : I^p \times I \longrightarrow \text{L-Imm}(D^k \times D^{n-k}, M)$ such that

i) $\tilde{F}(x, 0) = F_0(x)$,

ii) $\pi \circ \tilde{F} = F$.

In section 3 we sketch the proof of Theorem 2.1, given Theorem 2.2. We will prove Theorem 2.2 in section 4.

3. The immersion classification theorem

Let Λ be a smooth n-manifold and M be a contact smooth $(2n+1)$-manifold with contact structure σ.

We begin with a description of the set of Legendre planes in T(M), that is, the set of n-planes in the fibres of T(M) on which the contact form σ vanishes.

Lemma 3.1. Let M be orientable. The set L(M) of Legendre planes in T(M) has the structure of a bundle on M associated with T(M) and with fibre U(n)/O(n).

Proof. Consider a Euclidean space R^{2n+1} of dimension $2n+1$ with coordinate (x, y, z) $R^n \times R^n \times R$. The 1-form

$$\sigma = xdy + dz$$

$$= x_1dy_1 + \ldots + x_ndy_n + dz$$

defines a canonical contact structure on R^{2n+1}. The Legendre subspace of σ through the origin has equation $dz = 0$.

We take x and y as coordinates in this hyperplane. Therefore, in this plane we have

$$d\sigma\Big|_{\sigma=0} = dx \wedge dy. \qquad \text{(cf. Arnold [1])}$$

However, in the canonical symplectic 2n-space (R^{2n}, ω), $\omega = dx \wedge dy$, the set of lagrange planes is considered to be U(n)/O(n) (cf. Arnold [2], Souriau [12]). From this fact, we obtain the lemma.

Now suppose Λ has been given a Riemannian metric, and $\mathfrak{F}\Lambda \longrightarrow \Lambda$ be the frame bundle, i.e. principal $O(n)$-bundle associated with $T(\Lambda)$. Let $\mathfrak{F}L(M) \longrightarrow L(M)$ be the $O(n)$-bundle of n-frames in the Legendre planes in $T(M)$.

Corollary 3.2. L-monomophisms $\bar{\Phi} : T(\Lambda) \longrightarrow T(M)$ are in 1-1 correspondence with $O(n)$-bundle maps $\mathfrak{F}(\Lambda) \longrightarrow \mathfrak{F}L(M)$.

Theorem 3.3. The restriction map

$$\text{L-Mon}(T(D^k \times D^{n-k}), T(M)) \longrightarrow \text{L-Mon}(T(\partial D^k \times D^{n-k+1}), T(M))$$

is a fibration.

Proof. By Corollary 3.2, the covering homotopy theorem holds for L-monomorphisms. However, the assertion is simply a restatement of this property.

The next result is the 2nd step of the preparation for the proof of Theorem 2.1.

Lemma 3.4. Let $M = (M, \sigma)$ be a contact manifold of dimension $2n + 1$ and D^n be the n-disk. Then the map which maps the map f to its differential df

$$d : \text{L-Imm}(D^n, M) \longrightarrow \text{L-Mon}(T(D^n)), T(M))$$

is a homotopy equivalence.

Proof. Let 0 be the origin for D^n, and write L-Imm(0, M) for the germs of Legendre immersions at 0 of D^n into M. Let $r : D^n \longrightarrow D^n$ be a radial retraction of D^n onto a prescribed small neighborhood of 0, fixed on a smaller neighborhood of 0. By an argument formally identical to Haefliger-Poenaru [10],

$$r_* : \text{L-Imm}(D^n, M) \longrightarrow \text{L-Imm}(0, M)$$

is a homotopy equivalence.

On the other hand,

$$r_* : \text{L-Mon}(T(D^n), T(M)) \longrightarrow \text{L-Mon}(T_0(D^n), T(M))$$

is also a homotopy equivalence by Theorem 3.3. Since the diagram

$$\begin{array}{ccc} \text{L-Imm}(D^n, M) & \xrightarrow{d} & \text{L-Mon}(T(D^n), T(M)) \\ r_* \downarrow & & \downarrow r_* \\ \text{L-Imm}(0, M) & \xrightarrow{d} & \text{L-Mon}(T_0(D^n), T(M)) \end{array}$$

is commutative, it is sufficient to show that

$$d : \text{L-Imm}(0, M) \longrightarrow \text{L-Mon}(T_0(D^n), T(M))$$

is a homotopy equivalence.

However, an inverse $\text{L-Mon}(T_0(D^n), T(M)) \longrightarrow \text{L-Imm}(0, M)$ is porvided by Darboux's theorem (cf. Arnold [1], Appendix 4).

Proof of Theorem 2.1. By Theorem 2.2 (L-Imm($D^k \times D^{n-k}$, M), π, L-Imm($\partial D^k \times D^{n-k+1}$, M)) is a fibre space, where π is the restriction map. Furthermore, the following diagram :

$$\begin{array}{ccc} L\text{-}Imm(D^k \times D^{n-k}, M) & \xrightarrow{d} & L\text{-}Mon(T(D^k \times D^{n-k}), T(M)) \\ \pi \downarrow & & \pi_1 \downarrow \\ L\text{-}Imm(\partial D^k \times D^{n-k+1}, M) & \xrightarrow{d} & L\text{-}Mon(T(\partial D^k \times D^{n-k+1}), T(M)) \end{array}$$

is commutaive, namely d is a fibre map, where π_1 is the restriction map. Therefore, by Theorem 3.3 and Lemma 3.4 , we obtain Theorem 2.1, in formally identical method with Haefliger-Poenaru [10], Haefliger [8], [9].

4. Covering homotopy property for the space of Legendre immersions

Now we prove the covering homotopy property for the space of Legendre immersions into a compact regular contact manifold, i.e. Theorem 2.2.

Let $f_0 : I^p \longrightarrow L\text{-}Imm(D^k \times D^{n-k}, M)$, $f : I^p \times I \longrightarrow L\text{-}Imm(\partial D^k \times D^{n-k+1}, M)$ be continuous maps. Let

$$\pi : L\text{-}Imm(D^k \times D^{n-k}, M) \longrightarrow L\text{-}Imm(\partial D^k \times D^{n-k+1}, M)$$

be the map which maps g to the restriction $g|\partial D^k \times D^{n-k+1}$. Suppose $\pi \circ f_0(x) = f(x, 0)$. Then we want the lifting $\tilde{f}$ of f to $L\text{-}Imm(D^k \times D^{n-k}, M)$ with $\tilde{f}(x, 0) = f_0(x)$. Now M is a compact regular contact manifold, M is a principal S^1-bundle over a symplectic manifold B : (M, p, B), $B = (B, \omega)$. Moreover, we have $d\sigma = p^*\omega$. Corresponding to f_0, f, we have the following maps, respectively :

$$F_0 \; : I^p \times D^k \times D^{n-k} \longrightarrow M,$$

$$F \; : I^p \times I \times \partial D^k \times D^{n-k+1} \longrightarrow M.$$

Here, for each $(u, t) \in I^p \times I$, if we put $f_{u,t}(x) = F(u, t, x)$, $f_{u,0}(x) = F_0(u, x)$, $f_{u,t}$ are Legendre immersions. Composing these maps with $p : M \rightarrow B$, we have the following maps

$$G_0 : I^p \times D^k \times D^{n-k} \longrightarrow B,$$

$$G : I^p \times I \times \partial D^k \times D^{n-k+1} \longrightarrow B.$$

Here, if we put $g_{u,t}(x) = G(u, t, x)$, $g_{u,0}(x) = G_0(u, x)$, then $g_{u,t}$, $g_{u,0}$ are lagrange immersions, by Theorem 1.1.

Applying the flexibility theorem of lagrange immersions (cf. Gromov [7]), we have a family of lagrange immersions

$$\tilde{G} : I^p \times I \times D^k \times D^{n-k} \longrightarrow B,$$

which is an extension of both G_0 and G. However, for $k \neq 1$ by Theorem 1.1, we can lift $\tilde{G}$ to M, namely, we obtain the following C^∞-map

$$\tilde{F} : I^p \times I \times D^k \times D^{n-k} \longrightarrow M,$$

i) $\tilde{F}$ is an extension of both F_0 and F,

ii) if we put $\tilde{F}(u, t, x) = f_{u,t}(x)$, then $f_{u,t} : D^k \times D^{n-k} \longrightarrow M$ is a Legendre immersion, for each $(u,t) \in I^p \times I$,

iii) $p \circ \tilde{F} = \tilde{G}$.

Since we assume that the source manifold Λ is simply connected, we have obtained Theorem 2.2.

Proof of the existence of lift $\widetilde{F}$ for $k \neq 1$.

By taking sufficiently small cubic subdivision of $I^p \times I \times D^k \times D^{n-k}$, it suffices that we consider the case where the S^1-bundle (M, p, B) to be

$$M = (R^{2n+1}, \sigma), \qquad \sigma = \sum_i x_i dy_i + dz,$$

$$R^{2n+1} \ni (x_1, \ldots, x_n, y_1, \ldots, y_n, z)$$

$$B = (R^{2n}, \omega), \qquad \omega = d\underline{\sigma},$$

$$R^{2n} \ni (x_1, \ldots, x_n, y_1, \ldots, y_n),$$

$$\underline{\sigma} = \sum_i x_i dy_i,$$

$$p \; ; \; (x_1, \ldots, x_n, y_1, \ldots, y_n, z) \longmapsto (x_1, \ldots, x_n, y_1, \ldots, y_n)$$

Then for $(u, t) \in I^p \times I$, let

$$f_{u,t} : \partial D^k \times D^{n-k+1} \longrightarrow R^{2n+1}$$

$$F_{u,0} : D^k \times D^{n-k} \longrightarrow R^{2n+1}$$

be Legendre immersions with $F_{u,0}\big|\partial D^k \times D^{n-k+1} = f_{u,0}$. Let us denote as follows :

$$F_{u,0} = (X_{u,0}, Y_{u,0}, Z_{u,0}),$$

$$f_{u,t} = (x_{u,t}, y_{u,t}, z_{u,t}) \ ,$$

$$\Phi_{u,0} = (X_{u,0}, Y_{u,0}) = p \circ F_{u,0},$$

$$\varphi_{u,t} = (x_{u,t}, y_{u,t}) = p \circ f_{u,t}.$$

Then $\Phi_{u,0}$, $\varphi_{u,t}$ are lagrange immersions into (R^{2n}, ω) such that

$$(\varphi_{u,t})^* \sigma = -dz_{u,t},$$

(4.1)

$$(\Phi_{u,0})^* \sigma = - dZ_{u,0},$$

and $Z_{u,0} \big| \partial D^k \times D^{n-k+1} = z_{u,0}$.

(here we are considering $z_{u,t}$, $Z_{u,0}$ as coordinate functions on the bundle space on $\partial D^k \times D^{n-k+1}$ induced by $f_{u,t}$ and on $D^k \times D^{n-k}$ induced by $F_{u,0}$, respectively).

Assertion. In this situation, we have a lagrange immersion

$$\tilde{\varphi}_{u,t} : D^k \times D^{n-k} \longrightarrow (R^{2n}, \omega)$$

such that

$$\tilde{\varphi}_{u,t} \big| \partial D^k \times D^{n-k+1} = \varphi_{u,t},$$
$$\tilde{\varphi}_{u,0} = \Phi_{u,0}.$$

As is stated above, we use here the flexibility of lagrange immersions (cf. Gromov [7], Part III).

Now we construct a Legendre immersion $\tilde{F}_{u,t} : D^k \times D^{n-k} \longrightarrow R^{2n+1}$ with

$$\tilde{F}_{u,t}\Big|\partial D^k \times D^{n-k} = f_{u,t},$$

$\tilde{F}_{u,0} = F_{u,0}$. Since $\tilde{\varphi}_{u,t}$ is a lagrange immersion, we have $(\tilde{\varphi}_{u,t})^* \omega = 0$. Therefore, $(\varphi_{u,t})^* \underline{\sigma}$ is closed on $D^k \times D^{n-k}$. By Poincare's Lemma, we have a C^{∞}-function $K_{u,t} : D^k \times D^{n-k} \longrightarrow R$, such that

$$(\varphi_{u,t})^* \underline{\sigma} = -dK_{u,t}, \quad (u,\ t) \quad I^p \times I.$$

Suppose $k \geqq 2$. Then $\partial D^k \times D^{n-k+1}$ is connected. By (4.1) we have

$$K_{u,t}\Big|\partial D^k \times D^{n-k+1} = z_{u,t} + c_{u,t},$$

$$K_{u,0} = Z_{u,0} + C_{u,0},$$

here $c_{u,t}$ is a constant on $D^k \times D^{n-k+1}$ for each $(u,\ t) \in I^p \times I$, and $C_{u,0}$ is a constant on $D^k \times D^{n-k}$ for each $u \in I^p$. Then we have

$$\begin{aligned} C_{u,0}\Big|\partial D^k \times D^{n-k+1} &= K_{u,0}\Big|\partial D^k \times D^{n-k+1} - Z_{u,0}\Big|\partial D^k \times D^{n-k+1} \\ &= K_{u,0}\Big|\partial D^k \times D^{n-k+1} - z_{u,0} \\ &= c_{u,0}. \end{aligned}$$

Therefore, for each $(u,\ t) \in I^p \times I$, we can take a constant $\tilde{C}_{u,t}$

on $D^k \times D^{n-k}$ such that

0) $C_{u,t} \Big| \partial D^k \times D^{n-k+1} = c_{u,t}$,

$\widetilde{C}_{u,0} = c_{u,0}$,

1) $\widetilde{C}_{u,t}$ is smoothly dependent on $(u, t) \in I^p \times I$.

now we put

$$\widetilde{Z}_{u,t} = K_{u,t} - \widetilde{C}_{u,t} \ ;$$

$$\widetilde{Z}_{u,t} : D^k \times D^{n-k} \longrightarrow R, \quad \text{for} \quad (u, t) \in I^p \times I.$$

Then we have

$$Z_{u,t} \Big| \partial D^k \times D^{n-k+1} = z_{u,t},$$

$$\widetilde{Z}_{u,0} = z_{u,0}$$

We define for $(u, t) \in I^p \times I$

$$\widetilde{F}_{u,t} : D^k \times D^{n-k} \longrightarrow R^{2n+1},$$

$$\widetilde{F}_{u,t} = (\widetilde{\varphi}_{u,t}, \widetilde{Z}_{u,t}).$$

Then we have

$$(\widetilde{F}_{u,t})^* \sigma = (\widetilde{\varphi}_{u,t})^* \underline{\sigma} + d\widetilde{Z}_{u,t}$$

$$= 0,$$

namely, $\widetilde{F}_{u,t}$ is a Legendre immersion and

$$\tilde{F}_{u,t}\Big|\partial D^k \times D^{n-k+1} = f_{u,t},$$

$$\tilde{F}_{u,0} = F_{u,0}.$$

Thus we have a lift which we want.

Note. In case $k = 1$, $\tilde{C}_{u,t}$ as above os not well-defined.

Department of Mathematics
Faculty of Science
Kyoto University

Department of Mathematics
College of General Education
Osaka University

References

[1] V.I. Arnold, Mathematical Methods of Classical Mechanics, Springer, 1974

[2] V. I. Arnold, On a characteristic class which enters in quantization condition, Functional Analysis and its Appl.,1 , 1-14(1967).

[3] D. Bennequin, Entrlacements et equation de Pfaff, These Doctrat d'Etat, Univ. Paris VII, 1982.

[4] W. Boothby-H. C. Wang, On contact manifolds, Ann. of Math., 68, 721-734(1958).

[5] A. Douady, Noeds et structures de contact en dimension 3 [d'apres D. Bennequin], Seminaire Bourbaki, No. 604 (1983).

[6] M. L. Gromov, A topological technique for the construction of solution of differential equations and inequalities, Act. Congr. Intern. Math.,Vol. 2, 221-225(1970).

[7] M. L. Gromov, Partial Differential Relations, Springer~~(to appear)~~. 1986

[8] A. Haefliger, Homotopy and integrability, Lecture Notes in Math. Springer, 197, 133-163(1971).

[9] A. Haefliger, Lectures on the Theorem of Gromov, Lecture Notes in Math. Springer, 209, 128-141(1971).

[10] A. Haefliger-V. Poenaru, La classification des immersions combinatoires, Publ. Math. I.H.E.S., 23, 75-91(1964).

[11] J. A. Lees, On the classification of Lagrange immersions, Duke Math. J., 43217-224(1976).

[12] J. M. Souriau, Structure des systemes dynamiques, Dunot, Paris, 1970.

[13] N. Steenrod, The Topology of Fibre Bundles, Princeton Univ. Press, 1951.

1986. Dec